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ADVANTAGES OF CUBICS FOR APPROXIMATING ELEMENT BOUNDARIES

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Abstract—A finite element is constructed with two straight sides and one cubic side. A cubic isoparametric transformation chosen to satisfy particular curve-matching requirements while ensuring a positive transformation Jacobian over the element is investigated.

INTRODUCTION

Although the true shape of a curved boundary is often in doubt we feel that once it has been established either through an engineering drawing or a mathematical equation it should be altered as little as possible thereafter. In particular, special features of the curved boundary such as cusps, points of inflexion, etc., which have a major influence on the solution should be retained as far as possible. This is particularly so in fluids problems where the shape of the body dictates the entire flow pattern.

The present note investigates the isoparametric cubic transformation as a means of satisfying particular curve matching requirements whilst ensuring at the same time that the Jacobian of the transformation remains positive inside and on the boundary of the element.

LAGRANGE CUBIC ISOPARAMETRIC TRANSFORMATION

Regions with curved boundaries in *two space dimensions only* are considered in this note. A typical region is divided up using triangular elements each one of which inside the region has three straight sides, and adjacent to the boundary has two straight sides and one curved side. A typical boundary element is shown in Fig. 1(a) where the nodes on the straight sides are at the points of trisection and those on the curved sides are placed arbitrarily at the points (X_6, Y_6) and (X_7, X_7). The Lagrange cubic isoparametric transformation based on the nine nodes shown in Fig. 1 is given by

$$
\begin{aligned}
\begin{matrix} x(p,q) \\ y(p,q) \end{matrix} = {} & \left[\frac{1}{2}r(3r-1)(3r-2)-\frac{9}{2}pqr\right]^{x_1}_{y_1} + \left[\frac{1}{2}p(3p-1)(3p-2)-\frac{9}{2}pqr\right]^{x_2}_{y_2} \\
& + \left[\frac{1}{2}q(3q-1)(3q-2)-\frac{9}{2}pqr\right]^{x_3}_{y_3} + \left[\frac{9}{2}rp(3r-1)+\frac{27}{4}pqr\right]^{x_4}_{y_4} \\
& + \left[\frac{9}{2}rp(3p-1)+\frac{27}{4}pqr\right]^{x_5}_{y_5} + {-}{-}{-}{-}{-} + \left[\frac{9}{2}qr(3r-1)+\frac{27}{4}pqr\right]^{x_9}_{y_9},
\end{aligned}
\tag{1}
$$

where

$$r = 1 + p - q. \tag{2}$$

The interpolation formula on which the transformation (1) is based follows from Fig. 1(c). It has *second order* precision in p and q but lacks precision in all four cubic terms. Other second order interpolation formulae along with a discussion of the elimination of the internal tenth node, the latter being required for full cubic precision, is contained in[1]. Linear precision in x and y for the interpolant is guaranteed by the isoparametric transformation.

On the *curved side*, we have

$$p + q = 1, \qquad (r = 0) \tag{3}$$

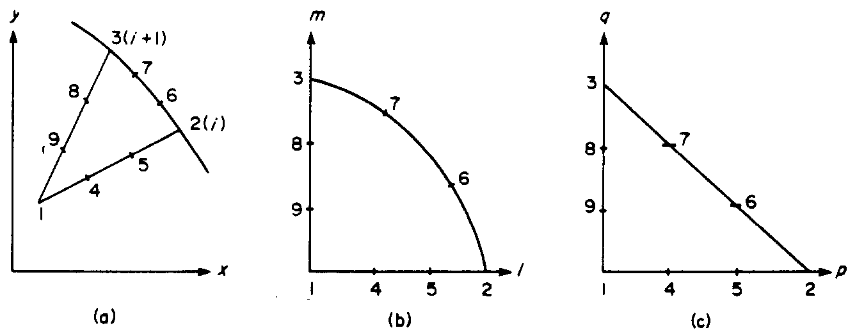

Fig. 1.

and so from (1) the parametric formulae for the curved side become

$$x(p,q) = x_3 + \left(x_2 - \frac{11}{2}x_3 - \frac{9}{2}x_6 + 9x_7\right)p - \frac{9}{2}(x_2 - 2x_3 - 4x_6 + 5x_7)\Big)\,p^2 + \frac{9}{2}(x_2 - x_3 - 3x_6 + 3x_7\Big)\,p^3$$

$$y(p,q) = y_3 + \left(y_2 - \frac{11}{2}y_3 - \frac{9}{2}y_6 + 9y_7\right)p - \frac{9}{2}\left(y_2 - 2y_3 - 4y_6 + 5y_7\right)p^2 + \frac{9}{2}(y_2 - y_3 - 3y_6 + 3y_7)p^3 \tag{4}$$

where $0 \le p \le 1$. We look upon the points (x_2, y_2), (x_3, y_3) as the fixed points (x_i, y_i), (x_{i+1}, y_{i+1}) and the points (x_6, y_6), (x_7, y_7) as the moveable points (X_6, Y_6), (X_7, Y_7), which we attempt to place to suit specific requirements. For example, if the slopes of the *original* curved side are f_i and f_{i+1} at the points (x_i, y_i) and (x_{i+1}, y_{i+1}) respectively and we wish the *implied* curve (4) to pick up these slopes exactly (required for cusps, points of inflexion) then it follows that the co-ordinates of the moveable points must satisfy the relationships

$$f_iX_6 - Y_6 - \frac{1}{2}f_iX_7 + \frac{1}{2}Y_7 = \frac{1}{9}\left[f_i\left(\frac{11}{2}x_i - x_{i+1}\right) - \left(\frac{11}{2}y_i - y_{i+1}\right)\right]$$

$$\frac{1}{2}f_{i+1}X_6 - \frac{1}{2}Y_6 - f_{i+1}X_7 + Y_7 = \frac{1}{9}\left[f_{i+1}\left(x_i - \frac{11}{2}x_{i+1}\right) - \left(y_i - \frac{11}{2}y_{i+1}\right)\right]. \tag{5}$$

THE JACOBIAN OF THE TRANSFORMATION

For convenience we now introduce the (l, m) plane (see Fig. 1(b)) and the transformation (1) is replaced by

$$\begin{aligned} x(l,m) &= x_1 - (x_1 - x_2)l + (x_3 + x_1)m \\ y(l,m) &= y_1 - (y_1 - y_2)l + (y_3 - y_1)m, \end{aligned} \tag{6a}$$

followed by

$$\begin{aligned} l(p,q) &= p + pq\left\{a + \frac{1}{2}b(p-q)\right\} \\ m(p,q) &= q + pq\left\{c - \frac{1}{2}d(p-q)\right\} \end{aligned} \tag{6b}$$

where

$$\begin{aligned} a &= \frac{9}{4}(L_6 + L_7 - 1), \qquad b = \frac{27}{2}\left(L_6 - L_7 - \frac{1}{3}\right) \\ c &= \frac{9}{4}(M_6 + M_7 - 1), \qquad d = \frac{27}{2}\left(-M_6 + M_7 - \frac{1}{3}\right), \end{aligned} \tag{7}$$

and the nodes (L_6, M_6) and (L_7, M_7) are the moveable points on the curved side in the (l, m) plane. The Jacobian of the transformation (1) can now be expressed in the form

$$\begin{aligned} J(x(p,q), y(p,q)) &= J(x(l,m), y(l,m))J(l(p,q), m(p,q)) \\ &= \Delta J(l(p,q), m(p,q)), \end{aligned}$$

where

$$\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} > 0,$$

and so

$$J(x(p, q), y(p, q)) > 0$$

for all points inside and on the boundary of the triangle if

$$J(l(p, q), m(p, q)) > 0.$$

This latter Jacobian is given by

$$J = 1 + cp + aq - \frac{1}{2}dp^2 - \frac{1}{2}bq^2 + (b + d)pq + \frac{1}{2}(ad + bc)pq(p + q), \tag{8}$$

and we require placements of the points (L_6, M_6) and (L_7, M_7) so that

$$J > 0 \tag{9}$$

for all positions of the point (p, q) in the standard triangle. General rules for the satisfaction of (9) are impractical and so we look at special cases of the cubic curve.

SPECIAL CASES OF THE CUBIC CURVE

(i) *The unique parabola.* It was shown in [2] that if

$$L_7 = L_6 - \frac{1}{3}, \qquad M_7 = M_6 + \frac{1}{3} \tag{10}$$

the cubic curve degenerates into the unique parabola passing through the four points (x_i, y_i), (X_6, Y_6), (X_7, Y_7) and (x_{i+1}, y_{i+1}). The moveable points in the (x, y) and (l, m) planes are connected by the linear transformation (6a) and the equation of the parabola in the (l, m) plane is

$$[2(cl - am) + (a - c)]^2 = (a + c)[(a + c) - 4(l + m - 1)]. \tag{11}$$

It follows from (10), (6a) and (5) that the parabola will have prescribed slopes f_i and f_{i+1} at its end points if the moveable points are located at

$$\begin{aligned} X_6 &= \left(\frac{2}{3}x_i + \frac{1}{3}x_{i+1}\right) + \frac{2}{9}\alpha, \qquad Y_6 = \left(\frac{2}{3}y_i + \frac{1}{3}y_{i+1}\right) + \frac{2}{9}\beta, \\ X_7 &= \left(\frac{1}{3}x_i + \frac{2}{3}x_{i+1}\right) + \frac{2}{9}\alpha, \qquad Y_7 = \left(\frac{1}{3}y_i + \frac{2}{3}y_{i+1}\right) + \frac{2}{9}\beta, \end{aligned} \tag{12}$$

where

$$\alpha = \frac{2(y_i - y_{i+1}) + (x_{i+1} - x_i)(f_i + f_{i+1})}{f_{i+1} - f_i}$$

and

$$\beta = \frac{2(x_{i+1} - x_i)f_i f_{i+1} + (y_i - y_{i+1})(f_i + f_{i+1})}{f_{i+1} - f_i}.$$

It is also shown in [4] that the Jacobian reduces to

$$J(p, q) = 1 + cp + aq,$$

and (9) will be satisfied provided the moveable points lie in the shaded region of Fig. 2.

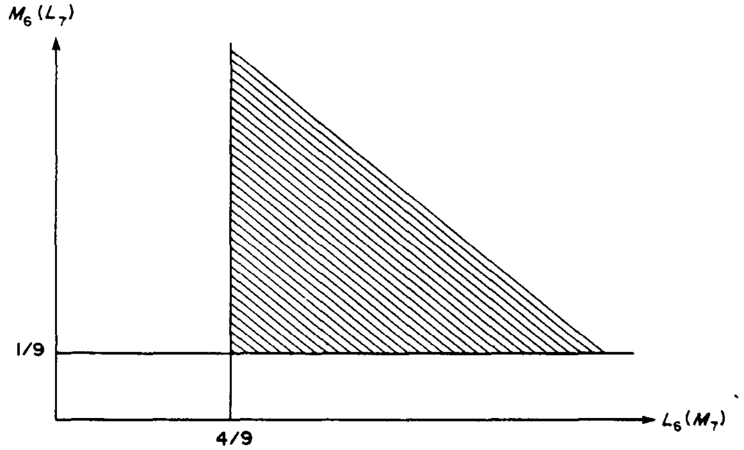

Fig. 2.

(ii) *The explicit cubic.* We now put

$$L_6 = \frac{2}{3}, \qquad L_7 = \frac{1}{3}, \tag{13}$$

and (6b) becomes

$$\begin{aligned} l &= p \\ m &= 1 - \frac{9}{2}\left(M_6 - 2M_7 + \frac{11}{9}\right) p + \frac{9}{2}(4M_6 - 5M_7 + 2)p^2 - \frac{9}{2}(3M_6 - 3M_7 + 1)p^3. \end{aligned} \tag{14}$$

Elimination of p in (14) leads to the explicit cubic curve

$$m = 1 - \frac{9}{2}\left(M_6 - 2M_7 + \frac{11}{9}\right) l + \frac{9}{2}(4M_6 - 5M_7 + 2)l^2 - \frac{9}{2}(3M_6 - 3M_7 + 1)l^3 \tag{15}$$

and contrasting examples of such curves are shown in Fig. 3, where in each case the l co-ordinates satisfy (13) and the m co-ordinates lie in the permissible region shown in Fig. 4. Such placements of the moveable points ensure $J > 0$. Also the explicit cubic curve will have prescribed slopes f_i and f_{i+1} at its end points in the (x, y) plane if the moveable points are located at

$$\begin{aligned} X_6 &= \frac{2}{3}x_i + \frac{1}{3}x_{i+1} + \frac{2}{27}(x_i - x_{i+1})(2X - Y) \\ Y_6 &= \frac{2}{3}y_i + \frac{1}{3}y_{i+1} + \frac{2}{27}(y_1 - y_{i+1})(2X - Y) \\ X_7 &= \frac{1}{3}x_i + \frac{2}{3}x_{i+1} + \frac{2}{27}(x_1 - x_{i+1})(X - 2Y) \\ Y_7 &= \frac{1}{3}y_i + \frac{2}{3}y_{i+1} + \frac{2}{27}(y_1 - y_{i+1})(X - 2Y) \end{aligned} \tag{16}$$

where

$$X = \frac{(x_i - x_{i+1})f_i - (y_i - y_{i+1})}{(x_1 - x_{i+1})f_i - (y_1 - y_{i+1})}$$

and

$$Y = \frac{(x_i - x_{i+1})f_{i+1} - (y_i - y_{i+1})}{(x_1 - x_{i+1})f_{i+1} - (y_1 - y_{i+1})}.$$

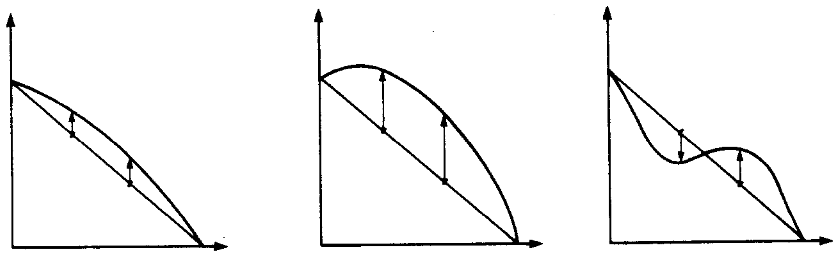

Fig. 3.

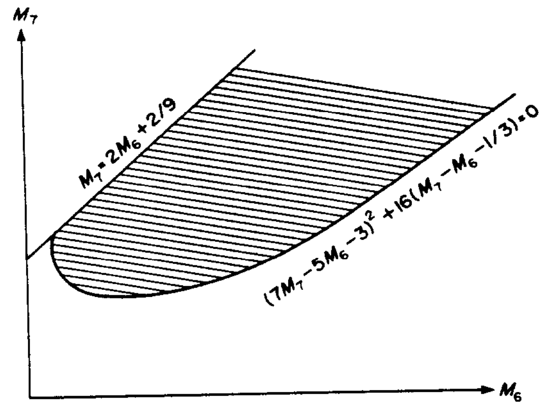

Fig. 4.

In a similar manner if we put

$$M_6 = \frac{1}{3}, \qquad M_7 = \frac{2}{3},$$

formula (6b) becomes

$$l = 1 - \frac{9}{2}\left(-2L_6 + L_7 + \frac{11}{9}\right) q + \frac{9}{2}(-5L_6 + 4L_7 + 2)q^2 - \frac{9}{2}(-3L_6 + 3L_7 + 1)q^3 \qquad (17)$$

$$m = q$$

and we obtain the other explicit cubic curve. This time examples of such curves are obtained by moving the points 6 and 7 parallel to the l-axis, ensuring of course that $J > 0$.

(iii) *The symmetric cubic.* This is obtained by symmetric placement of the points 6 and 7 about the line $m = l$, that is

$$M_6 = L_7 \text{ and } M_7 = L_6. \qquad (17)$$

Its equations in the (l, m) plane is

$$4a^3(l - m)^2 = [2a + b(l + m - 1)]^2(a - 2(l + m - 1)], \qquad (18)$$

and $J > 0$ provided L_6 and L_7 lie in the shaded region in Fig. 5. This cubic curve is similar to the one used by Williams and Morton[3] in the calculating of compressible flow past a circular cylinder for a range of Mach numbers.

All the results in this paper so far have been based on the Lagrange cubic interpolant. They could equally well have been obtained using the Hermite cubic interpolant. The formulae connecting the parameters in these two interpolants can be found in[4].

CUBIC PRECISION

A crucial factor affecting the accuracy and rate of convergence in the finite element method is the *maximum* degree of polynomial spanned by the basis functions *in the* (x, y) *plane.* It has

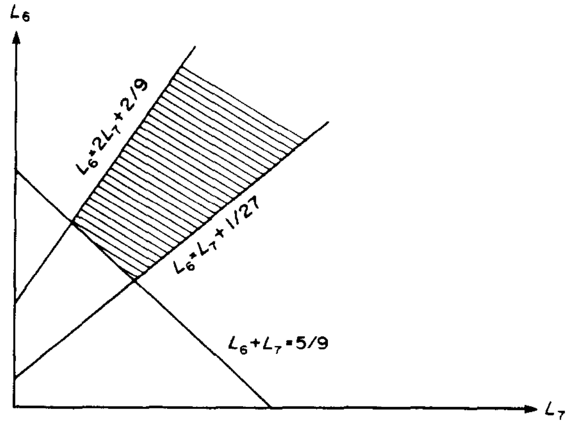

Fig. 5.

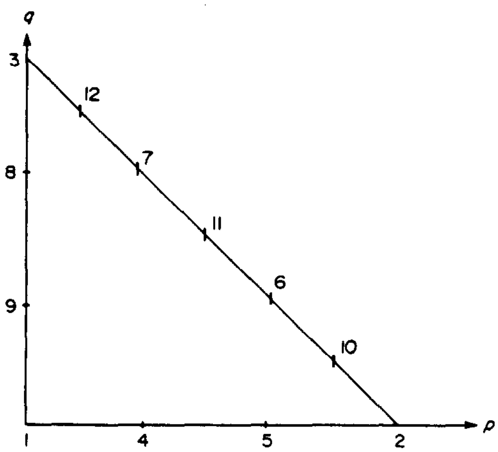

Fig. 6.

already been stated that any isoparametric transformation guarantees *linear* precision for the interpolant in the (x, y) plane. McLeod[5] has shown that cubic precision can be obtained in the (x, y) plane provided three extra nodes are considered on the *implied* curved side, (see Fig. 6) and the latter is a *conic*, (see (1) the unique parabola).

These nodes are numbered 10, 11, and 12, and are located on the slant side $p + q = 1$ at the positions (1/6), (1/2), and (5/6) along the side respectively. The basis functions $W_i(i = 1, 2, \ldots, 12)$ which span polynomials of degree *three* in the (x, y) plane are given by

$$W_4 = -\frac{9}{2}p(1-p-q)(3p+3q-2)$$

$$W_9 = -\frac{9}{2}q(1-p-q)(3p+3q-2)$$

$$W_i(x, y, p, q) = C_i(x, y) - C_i(x_4, y_4)W_4 - C_i(x_9, y_9)W_9 \qquad i \in I$$

where $\{C_i(x, y)\}$, $i \in I$ is a unique set of linearly independent cubic polynomials with the property

$$C_i(x_j, y_j) = \delta_{ij} \qquad i, j \in I.$$

The set I contains the numbers 1 to 12 less 4 and 9.

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